Harj. 4.

1. $$\begin{cases} 2x - y - z = 3 \\ 3x + 2y + z = 4 \\ x - y - 2z = 5 \end{cases} \quad (1),\ (2)$$

$$\begin{cases} 5x + y = 7 \\ 3x + 2y + z = 4 \\ 7x + 3y = 13 \end{cases} \quad (-3)$$

$$\begin{cases} (1) & 5x + y = 7 \\ (2) & 3x + 2y + z = 4 \\ (3) & -8x = -8 \end{cases} \quad \overset{(3)}{\therefore} x = 1 \overset{(1)}{\therefore} y = 2 \overset{(2)}{\therefore} z = -3$$

2. $$\underbrace{\begin{bmatrix} 2 & -1 & -1 \\ 3 & 2 & 1 \\ 1 & -1 & -2 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}}_{=B}$$

(a) kehitys 1. rivin suht.
(b) 2x2-muistisääntö

$$\det A = \begin{vmatrix} 2 & -1 & -1 \\ 3 & 2 & 1 \\ 1 & -1 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & -1 \\ 5 & 1 & 1 \\ -3 & 1 & -2 \end{vmatrix} \overset{(a)}{=} -1 \begin{vmatrix} 5 & 1 \\ -3 & 1 \end{vmatrix} \overset{(b)}{=} -1(5+3) = -8$$

(-1) (2)

$\det A \neq 0$ $\therefore$ Cramérin sääntöä voi soveltaa

$$\det A_1 = \begin{array}{c} \begin{array}{ccc} B & & \\ \downarrow & & \end{array} \\ \begin{vmatrix} 3 & -1 & -1 \\ 4 & 2 & 1 \\ 5 & -1 & -2 \end{vmatrix} \end{array} = \begin{vmatrix} 0 & 0 & -1 \\ 7 & 1 & 1 \\ -1 & 1 & -2 \end{vmatrix} = \dots = -8$$

(-1)

(3)

$$\det A_2 = \begin{array}{c} \begin{array}{ccc} & B & \\ & \downarrow & \end{array} \\ \begin{vmatrix} 2 & 3 & -1 \\ 3 & 4 & 1 \\ 1 & 5 & -2 \end{vmatrix} \end{array} \overset{(a)}{=} 2\begin{vmatrix} 4 & 1 \\ 5 & -2 \end{vmatrix} - 3\begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix} - \begin{vmatrix} 3 & 4 \\ 1 & 5 \end{vmatrix}$$

$$\overset{(b)}{=} 2\cdot(-13) - 3\cdot(-7) - 11 = -16$$

$$\det A_3 = \begin{array}{c} \begin{array}{ccc} & & B \\ & & \downarrow \end{array} \\ \begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & 4 \\ 1 & -1 & 5 \end{vmatrix} \end{array} \overset{(a)}{=} \dots = 2\cdot 14 + 1\cdot 11 + 3(-5) = 24$$

Siis

$$x = \frac{\det A_1}{\det A} = \frac{-8}{-8} = 1$$

$$y = \frac{\det A_2}{\det A} = \frac{-16}{-8} = 2$$

$$z = \frac{\det A_3}{\det A} = \frac{24}{-8} = -3$$

4.

$$\begin{cases} 3x + y + z = 3 \\ x + 2y + z = 4 \end{cases} \quad (-1)$$

$$\begin{cases} 3x + y + z = 3 \\ -2x + y = 1 \end{cases} \quad (-1)$$

$$\begin{cases} 5x + z = 2 \\ -2x + y = 1 \end{cases}$$

Olk. $x = t$. Silloin

$$\begin{cases} x = t \\ y = 1 + 2t \\ z = 2 - 5t \end{cases}, \quad t \in \mathbb{R}$$

5.

$A\underline{X} = \underline{0}$ on ratkeava (esim $\underline{X} = \underline{0}$)

Ratkaisun parametrien lkm on

$$\underbrace{n}_{=3} - \operatorname{rank} A.$$

$\operatorname{rank} A \overset{?}{=} 3$?

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{vmatrix} (-1) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 3 & 3 \end{vmatrix} (-3)$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 0 & 0 & 0 \end{vmatrix} \overset{\text{nollarivi}}{=} 0$$

Siis $\operatorname{rank} A < 3$

rank $A = 2$?

löytyy 2×2-alimatr. jonka det on $\neq 0$

$$\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1\cdot 3 - 1\cdot 2 = 1 \neq 0$$

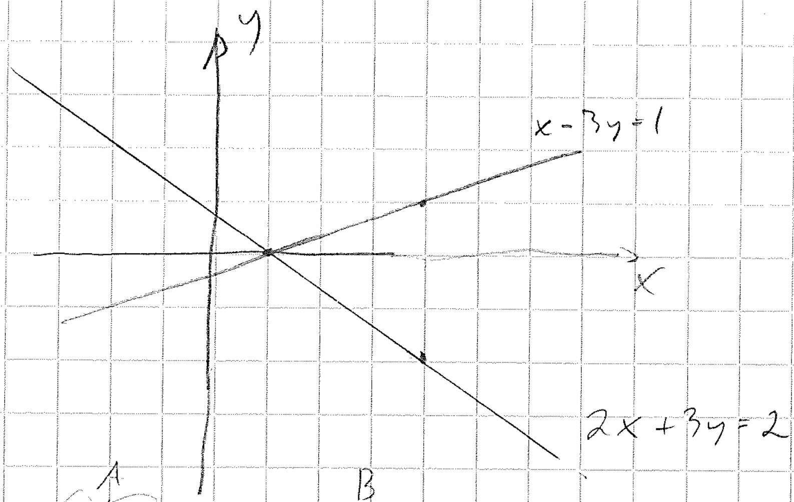
Siis rank $A = 2$.

Param. lkm on

$$n - \text{rank}\, A = 3 - 2 = 1.$$

6.

a)



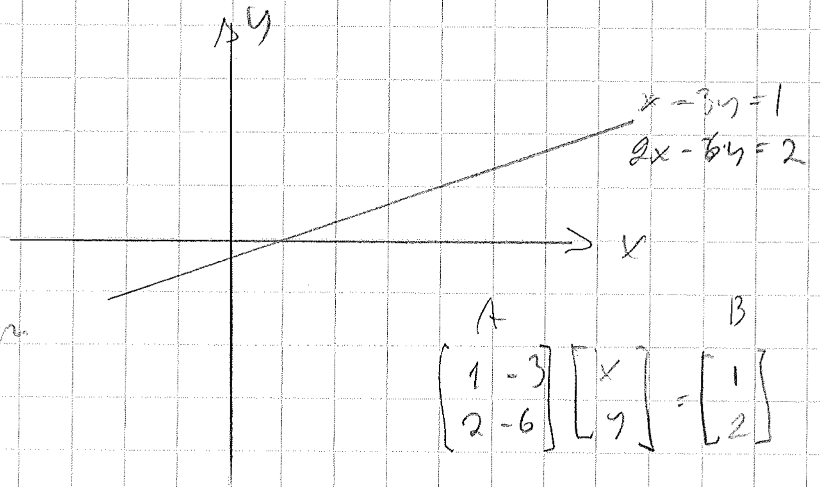
yksi ratk

$$\overbrace{\begin{bmatrix} 1 & -3 \\ 2 & 3 \end{bmatrix}}^{A} \begin{bmatrix} x \\ y \end{bmatrix} = \overbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}^{B}$$

$\det A = 9 \neq 0$

rank $A = 2$
rank $[A|B] = 2$ $\therefore$ ratkeava
param lkm
$2 - \operatorname{rank} A = 0$

b)



ratkaisuja äärettömän määrä

$$\overset{A}{\begin{bmatrix} 1 & -3 \\ 2 & -6 \end{bmatrix}} \begin{bmatrix} x \\ y \end{bmatrix} = \overset{B}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

$\det A = 0$

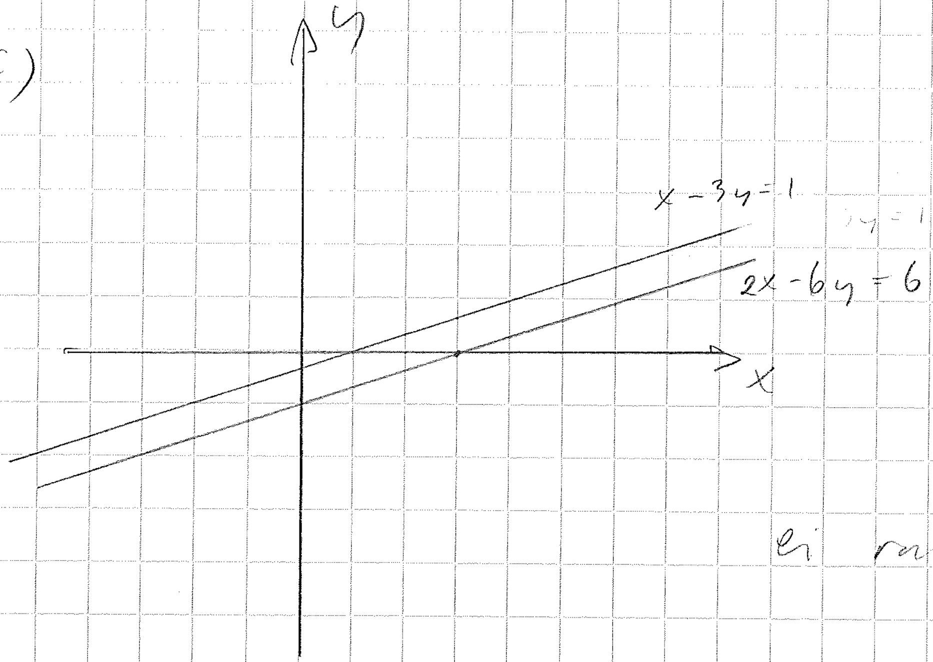
rank $A = 1$
rank $[A|B] = 1$

$\therefore$ ratkeava

parametrien lkm

$2 - \operatorname{rank} A = 1$

c)



ei ratkaisuja

$$\begin{array}{ccc} A & & B \\ \begin{bmatrix} 1 & -3 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} & = & \begin{bmatrix} 1 \\ 6 \end{bmatrix} \end{array}$$

$\det A = 0$

$$\left(\begin{array}{l} \text{rank } A = 1 \\ \text{rank } [A|B] = 2 \end{array} \neq \quad \therefore \text{ ei ratkaisua} \right)$$

7. a)

Väite: Kaikilla $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ on voimassa $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$.
Tod. Olkoot $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ Nyt

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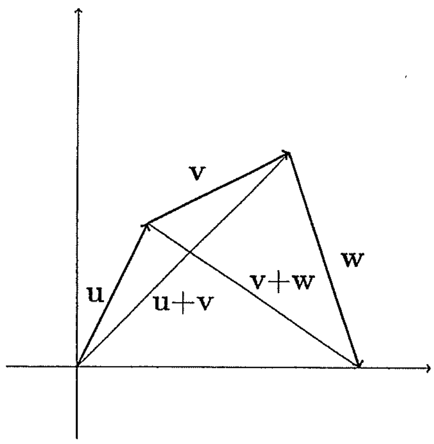
$$
\begin{aligned}
&(\mathbf{u}+\mathbf{v})+\mathbf{w} \\
&= \big((u_1, \ldots u_n) + (v_1, \ldots, v_n)\big) + (w_1, \ldots, w_n) && \text{[merkintä]} \\
&= (u_1 + v_1, \ldots, u_n + v_n) + (w_1, \ldots, w_n) && \text{[summan määr.]} \\
&= ((u_1 + v_1) + w_1, \ldots, (u_n + v_n) + w_n) && \text{[summan määr.]} \\
&= (u_1 + (v_1 + w_1), \ldots, u_n + (v_n + w_n)) && \text{[reaalialgebra]} \\
&= (u_1, \ldots, u_n) + (v_1 + w_1, \ldots, v_n + w_n) && \text{[summan määr.]} \\
&= (u_1, \ldots, u_n) + \big((v_1, \ldots, v_n) + (w_1, \ldots, w_n)\big) && \text{[summan määr.]} \\
&= \mathbf{u} + (\mathbf{v}+\mathbf{w}). && \text{[merkintä]}
\end{aligned}
$$

b)



c) Väite: $-\mathbf{u} = (-u_1, -u_2, \ldots, -u_n)$, kun $\mathbf{u} \in \mathbb{R}^n$.
Tod. Olkoon $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n$ ja merkitään $\mathbf{v} = (-u_1, \ldots, -u_n)$ Nyt

$$
\begin{aligned}
\mathbf{u}+\mathbf{v} &= (u_1, \ldots, u_n) + (-u_1, \ldots, -u_n) && \text{[merkintä]} \\
&= (u_1 + (-u_1), \ldots u_n + (-u_n)) && \text{[summan määr.]} \\
&= (0, \ldots, 0) && \text{[vastaluvun määr.]} \\
&= \mathbf{0} && \text{[nollavektorin määr.]}
\end{aligned}
$$

Siis $-\mathbf{u} = \mathbf{v} = (-u_1, \ldots, -u_n)$.

8.

a) Yhtälön $A\underline{X} = B$ ratkaisu on

(*) $\underline{X} = \underline{X}_p + \overline{\underline{X}}_h.$

Tod.

1) Tod, että (*) toteuttaa yhtälön:

$A(\underline{X}_p + \underline{X}_h)$

$= \underbrace{A\underline{X}_p}_{=B} + \underbrace{A\underline{X}_h}_{=\underline{0}}$ osittelu

$= B$

Siis (*) toteuttaa yhtälön.

2) Tod. että kaikki ratkaisut ovat tässä:

Olk. $\underline{X}$ mielivalt. ratkaisu

$\underline{X}_p$ yksi ratkaisu

Silloin

$A(\underline{X} - \underline{X}_p) = A\underline{X} - A\underline{X}_p = \underline{0} - \underline{0} = \underline{0}$

Siis $\underline{X} - \underline{X}_p$ on homog. yhtälön ratkaisu eli

$\underline{X} - \underline{X}_p = \underline{X}_h$

t.s. $\underline{X} = \underline{X}_p + \underline{X}_h$

8b)

$$\underline{x} = \begin{bmatrix} t \\ 1+2t \\ 2-5t \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}_{=\underline{x}_p} + t \underbrace{\begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}}_{=\underline{x}_h} \qquad t \in \mathbb{R}$$